Structural view independence: A criterion for judging the objectivity of economic parameters measured by opinion survey

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Summary

Economic measurements have a great influence over all our lives, but as with other soft measurements, significant effort is needed to ensure their objectivity. This is particularly true of the process of consolidating into a single representative figure different people's valuations of a non-market good ("views"), as measured by opinion survey. Sometimes the views are transformed and averaged before being returned to the original domain as the back-transformed mean. Examples are the geometric mean resulting from a logarithmic transformation and the root-mean-squared (r.m.s.) value from a square transformation. Such transformations are tested for objectivity using the new criterion of structural view independence. It is shown that an analyst using any general, nonlinear, increasing and differentiable transformation other than the linear transformation can know at the outset that he is giving greater weight to views of his choosing, meaning that he has no claim to objectivity. Of all such transformations, only the linear transformation possesses the desirable property of structural view independence. The resultant sample mean is objective and the only consolidated figure recommended for human views.

Highlights:

1. It is proved that the back-transformed mean of a power transformation is strictly increasing in the power, negative powers included. This result extends Cauchy's theorem that the geometric mean (zero power transformation) must be less than the sample mean (unity power transformation).

2. A general proof is given that the back-transformed mean of a nonlinear, increasing transformation that is concave will be less than the sample mean, while that of a nonlinear, increasing transformation that is convex will be greater than the sample mean.

3. Structural view independence is introduced as a new criterion for judging the objectivity of the process for consolidating human valuations into a single figure.

4. Buckingham's Pi theorem is used to show that if the transformation does not possess structural view independence, then each human view is filtered by the view of at least one other person in the consolidation process.

5. An analyst choosing any strictly nonlinear, increasing and differentiable transformation will be deciding in advance whether the back-transformed mean he will report is less than or greater than the sample mean.

6. Trimmed means, including the median, are shown to violate the criterion of structural view independence and should not be used to consolidate human views.

7. Examples are provided to illustrate the new theory, culminating in the simulation of opinion development over time within a focus group.

Key words: Structural view independence; Sample mean; Opinion survey; Soft measurement; Economic measurement; Contingent valuation.

1. Introduction

In a very real sense, our understanding of the way the world works relies on measurement. The interpretation of the precise measurements possible in the natural sciences has enabled powerful and convincing theories to be developed to explain the physical universe. Moreover, early success in the sphere of physical science has been backed up by the more recent development of a sound philosophical and theoretical basis for the measurement process itself (Finkelstein, 1994). But some measurements with great importance for the way in which we live our lives lie outside the realm of physical science, particularly measurements in economics and the decision sciences, used by Government and others to guide policy.

The potential influence of such economic measurements is illustrated vividly in J. K. Galbraith's description of U.S. production planning early in the Second World War (Galbraith, 1987, 1989), when economist, Robert Nathan

> "outlined a schedule of weapons production – aircraft, tanks, ordnance, ships – called the Victory Program. It far exceeded anything that others in Washington ... thought possible, even sane. But there were the tables; they showed how great were the unused and available resources. The Victory Program was adopted and with no undue difficulty achieved."

The tables cited were taken from the measurements of U.S. Gross National Product. Germany, by contrast, had no such national accounts, and "In the absence of knowledge of how the resources were being used, civilian consumption and man- and womanpower use in the civilian sector remained uncontrollably high throughout nearly the whole of the war" (*op. cit.*). Today, quarterly measurements of Gross Domestic Product are routinely pored over by economic analysts in all sectors of developed economies and provide the basis for policy setting by governments.

But despite their influence and potential power, economic measurements have difficulties that are generally greater than those of many physical measurements. Indeed, Finkelstein (2005) has considered the particular challenges associated with measuring national income in the wider context of the problems of measurement in soft systems. As noted by Boumans (2005), economic systems may be characterised formally as "soft", since they satisfy two out of the three Finkelstein criteria for being regarded as such: (i) they involve human action, perception, feeling, decisions and the like and (ii) they usually have significant size and complexity, making experimental determination of relations between system elements impractical.

The operation of an economic free market leads to a monetary value being placed on a good, in which case the measurement task is reduced simply to reading off the price at which the transaction between buyer and seller was carried out. However, there are and always will be a large number of instances where no obvious market exists, particularly in the case of public goods (e.g. clean air, the continued survival of a rare species of plant and so on), for which nevertheless it is desirable to know the value that people place on an amenity so that a cost-benefit analysis may be carried out. "Revealed preferences" may enable a measurement of the value of the non-market good. Here the good's value is determined by observing what quantifiable benefit has been given up in order to secure a certain quantity of that good or else, if the good is undesirable, what quantifiable burden has been taken on to avoid it. Clearly, the revealed preference technique requires a model of the system, and, as a minimum, that model needs to be transparent to a person possessed of the necessary mathematical skills.

As an alternative, attempts are sometimes made to determine the value of a non-market good using opinion polling or market-survey techniques. These "stated preference" methods may on occasion be referred to as "contingent valuation" (Ciriacy-Wantrup, 1947). Opinion polls and market surveys are based on the notion that an opinion that is in some sense representative of an entire population can be deduced fairly accurately from the judgements of a sample of people drawn from that population.

For example, a number of researchers have tried to measure the public's subjective valuation of safety measures. Thus, for example, Viscusi et al. (1987) attempted to measure how much the public would be willing to pay to reduce the frequency of human harm by various amounts. Meanwhile in the UK, opinion surveys have been used to justify a quantity termed the "value of a prevented fatality" (VPF) or the "value of a statistical life" (VSL) (Beattie *et al.*, 1998, Carthy *et al.*, 1999). Although there are difficulties associated with what this parameter actually represents (see Thomas *et al.*, 2009), such approaches have been cited as the basis of the safety cost-benefit approach recommended by the UK Government in the transport sector and other sectors (Department for Transport, 2009, 2012). Other surveys have attempted to find out, given that an individual is going to die at a specified time, if some ways of dying are worse than others – so-called "dread risks" (e.g. Chilton *et al.*, 2006).

At this point it is helpful to recall the position of measurement theorists, Mari *et al.* (2009) who regard objectivity and inter-subjective testability as the critical features for the reliability and dependability of measurement. Finkelstein takes a similar stance, stating that "measurement owes its power, primarily, to the objectivity of its description" (Finkelstein, 2005).

Lack of objectivity may arise from the features of the model necessary to interpret the raw measurement data and provide information on the property actually of interest, of which the measurand is normally only a manifestation. Indeed Finkelstein observes (*ibid.*) that the observer/analyst in social sciences has been argued not to be objective, but to operate "on the basis of ideologically motivated theories". It is not surprising, therefore, that Mari and Ugazio (2010), should see the validation of soft measurements as a topic of increasing importance, and hold that "the meaningfulness ... of the obtained results depends on interpretive models, that become pivotal for the validation of measurements and results". It is clear, given the clear potential for subjectivity and the widely held suspicions thereof, that validation of the measurement process is particularly important when measuring the behaviour of soft systems.

This paper will be concerned with the measurement of opinion. In particular, it will examine the interpretation of the results of an opinion poll in which people are asked to assess the value of a given good, where the value is measured on a continuous scale, often in money terms. As a matter of terminology, in this paper each person's numerical opinion will be referred to as a "view", and assumed to be non-negative.

The particular focus of the paper will be on the process by which different people's judgements are consolidated into a single figure which is then taken to be representative in some sense of the population as a whole. The paper will examine the objectivity inherent in the algorithm used to perform the consolidation into a single figure of the various views in the sample.

While the sample mean might appear to be the obvious candidate as a central measure, a different procedure is sometimes used. A nonlinear function is applied to transform the parameter of interest into a new variable. The mean of the new variable is found and then the inverse function applied to back-transform the resultant value onto the line of the original parameter, so that it has the same dimension. The paper will examine whether and when the very process of transformation, averaging and back-transformation, introduces subjectivity into the measurement process, and how this may be avoided.

It needs to be remembered that there are some basic differences in the way that the measurements of physical properties and human opinions can be consolidated. In the case of physics, theory may exist to suggest that the physical quantity has one unique value so that all measurements of the same parameter ought to give a similar result. Thus when outliers are detected in repeated physical measurements, it is legitimate for suspicion to fall on the measurement process – calibration errors, malfunctioning equipment and so on. Should a fault be found with the process by which an outlier has been generated, then it is fair to reject that particular physical measurement. Importantly, even if the source of the difference cannot be understood in detail, at least a conceptual model is in place to provide some degree of legitimacy for reducing the weighting given to certain data points or, indeed, for rejecting them altogether.

The situation with regard to human opinion is different. In many if not most cases, no convincing theory may exist to state that the parameter to be measured by survey should have a unique value, so that there is no *a priori* reason for all the views gathered being close to each other. Thus it is not reasonable to expect the treatment of outliers sometimes used in interpreting repeated physical measurements to transfer unchanged to the analysis of opinion surveys. In particular, once procedural errors, such as mistakes in transcription, have been corrected, no basis exists for rejecting a view or according it a reduced weighting simply because it is significantly different from the rest. Such a procedure would be justified only if it could be proved that the person was not qualified to offer an opinion, independent of the view offered. Such cases can be envisaged – for example the person might be insane, or too young to have acquired necessary experience, or inexpert in the field in which competence was required, or deliberately attempting to subvert the survey. But such cases are likely to have been screened out already in the process of attempting to find a representative sample of the population of interest.

Fundamentally, in the absence of a theory to suggest that opinions should cluster together, it needs to be accepted that it is legitimate for human views to differ markedly. An "equal rights" philosophy needs to be adopted when interpreting opinion survey data, should a survey be chosen as the method for determining the value of the good.

The equal rights philosophy just outlined is taken as axiomatic in the remainder of the paper. The methods of mathematics are then used to explore the implications and prove the results. It has to be admitted that some rather tight mathematical arguments are brought to bear, but a summary will be given at the beginning of each section to explain its direction and purpose to promote ease of understanding.

Many of the transformations currently used in analysing opinion poll data are power transformations. Both the root-mean-square and the geometric mean are in this category, as indeed is the arithmetic mean. (Appendix A explains how the geometric mean falls into this categorisation). Their prominence and importance means that it is sensible to analyse power transformations first. Thus Section 2 will show that it is possible to rank the sizes of the back-transformed mean, Z, according to the power used in the transformation. The treatment generalises to all power transformations the result first proved by Cauchy (1821), that the geometric mean must be less than the arithmetic or sample mean. The section ends with an example of an opinion poll where people are asked to place a value on the average total of life-years to come, the "value of human life".

Section 3 will then extend the approach to general, nonlinear transformations that are strictly increasing functions of their argument. It will be shown that the size of the back-transformed mean relative to the sample mean will depend on whether the nonlinear transformation is concave or convex. An illustration of a nonlinear transformation that is not a power transformation is given at the end of the section.

Section 4 introduces and explains the new criterion of structural view independence, which is implied by the equal rights philosophy. It will be shown that the sensitivity to all views of the back-transformed mean must be the same for structural view independence to occur.

Section 5 quantifies the result of the previous section, showing that the sensitivity of the back-transformed mean to each view must be equal to the inverse of the number of people in the sample for a transformation exhibiting structural view independence. The linear transformation satisfies this condition and produces a back-transformed mean equal to the sample mean.

Section 6 uses dimensional analysis to explore the implications of structural view dependence. It is proved that of all the nonlinear, increasing and differentiable transformations, the only one that possesses structural view independence is the linear transformation. No other transformation from this broad category is consistent with the equal rights philosophy necessary for the objective interpretation of opinion survey results.

Section 7 examines the structural view dependence and independence of one family of transformations used extensively in data analysis, namely the power transformations. In addition to providing corroboration for the more general analysis of Section 6, it demonstrates how an analyst can guarantee that his back-transformed mean will be less than the sample mean if he chooses a power lower than the value of unity associated with the linear transformation and the sample mean. Moreover, it shows that this result will have been achieved by systematically downgrading the importance given to people with high views. The converse will apply if the analyst chooses a power greater than unity.

Section 8 tests the objectivity of double sided, trimmed means, including the median, using the criterion of structural view independence. It is concluded that these should not be used to consolidate human views. A discussion will be given of the culpable manipulation of the London Inter-Bank Offer Rate (LIBOR), calculated as the trimmed mean of an opinion survey of bankers. It will be highlighted how problematic it is to use an opinion survey as the basis for an important economic parameter when all respondents have an incentive to falsify their views.

Section 9 gives a numerical example based on the behaviour of a focus group tasked with estimating the worth to the public of an environmental protection system. It shows views may evolve over time and the effect this has on the geometric mean. The view of one person comes to dominate the process not because he is the most knowledgeable or expert, but simply because his view is the lowest.

Section 10 sets out the conclusions.

It will be shown that some methods of data consolidation, possibly valid when applied to physical measurements, may have no validity when applied to soft measurements, specifically when the soft measurement is a human opinion.

2. Power transformations

The root-mean-squared (r.m.s.) value is an example of a power transformation. It is found by squaring the measurements, dividing their sum by their number and then taking the square root. Thus the power of the transformation is 2 in this case. A similar procedure is applied routinely in electrical engineering to the continuous measurement of alternating current, where the mean value will be zero. The r.m.s. value will be non-zero, and it has a real, physical significance, allowing the power produced in a resistive circuit to be calculated easily, for example.

Another power transformation may be made by taking logs, averaging the logarithmic values and then exponentiating to give the geometric mean. Appendix A explains how the logarithmic transformation is equivalent to a power transformation with a power that tends to zero.

Both the transformations just cited are instances of power transformations of the form:

$$Z = \left(\frac{1}{N} \sum_{i=1}^{N} X_i^b \right)^{\frac{1}{b}} \qquad (1)$$

where X_i is the numerical opinion or view of person, i, treated as a random variable, b is some power, N is the size of the sample and Z is the back-transformed mean. Putting the power, b, equal to 1.0 in equation (1) will, of course, give the sample mean. Thus the sample mean is also an instance of a power transformation. To cite a further example, it would be possible to find the squared-mean-root value:

$$Z = \left(\frac{1}{N} \sum_{i=1}^{N} \sqrt{X_i} \right)^2 \qquad (2)$$

Each of these central estimates will give a different value, so the question arises as to which is valid, and on what basis. This section will begin the investigation of this question. On the way, it will generalise to all power transformations and back-transformations Cauchy's result that the geometric mean is always less than the sample mean, and back this up with a numerical example.

2.1 Analysis

Let the power transformation, $h(X)$, be defined by

$$Y = h(X) = aX^b + c \qquad (3)$$

where *a, b* and *c* are constants and *Y* is a new random variable. We will impose the convention that $a > 0$, so that *Y* is increasing in *X* for $b > 0$. The inverse function is given by:

$$X = h^{-1}(Y) = \left(\frac{Y - c}{a} \right)^{\frac{1}{b}} \qquad (4)$$

The process of averaging the new variable, *Y*, defined by equation (3) gives the mean of the transformed variable, *W*, as:

$$W = \frac{1}{N} \sum_{i=1}^{N} Y_i = \frac{a}{N} \sum_{i=1}^{N} X_i^b + c \qquad (5)$$

The backward transformation gives the back-transformed mean, *Z*:

$$\begin{aligned} Z &= \left(\frac{W - c}{a} \right)^{\frac{1}{b}} \\ &= \left(\frac{1}{N} \sum_{i=1}^{N} X_i^b \right)^{\frac{1}{b}} \end{aligned} \qquad (6)$$

It will be observed from the final line of equation (6) that the process of transformation and back-transformation renders the values of the terms *a* and *c* irrelevant, so that we may replace them by $a = 1$ and $c = 0$ without loss of generality for all instances except when $b \to 0$, when it is necessary to set $a = b^{-1}$ and $c = -b^{-1}$ (see Appendix A , which discusses the log transformation valid as the limiting case when b is very small). Thus the simpler transformation:

$$Y = X^b \qquad (7)$$

is representative of the apparently more complex transformation of equation (3). [Interestingly, the transformation, $Y = X^b$, is decreasing in X for $b < 0$, approaching zero as $X \to \infty$, while the transformation, $Y = b^{-1}(X^b - 1)$ is increasing in X for $b < 0$, approaching the (positive) asymptote $-b^{-1}$ as $X \to \infty$. Nevertheless from equation (6), both transforms give the same value for the back-transformed mean.]

Using equations (5) and (6) when $a = 1$ and $c = 0$, the mean of the transformed variable is

$$W = \frac{1}{N}\sum_{i=1}^{N} Y_i = \frac{1}{N}\sum_{i=1}^{N} X_i^b \qquad (8)$$

while the back-transformed mean is:

$$Z = W^{\frac{1}{b}} \qquad (9)$$

It is possible to deduce the effect of changes in b by differentiation, the stages of which are given in equations (10) and (11) below. Differentiating equation (8) with respect to b gives:

$$\frac{dW}{db} = \frac{1}{N}\sum_{i=1}^{N} X_i^b \ln X_i \qquad (10)$$

Meanwhile, using the substitution, $r = 1/b$, in equation (9), so that $Z = W^r$ allows us to progress via:

$$\frac{dZ}{db} = \frac{\partial Z}{\partial W}\frac{dW}{db} + \frac{\partial Z}{\partial r}\frac{dr}{db} \qquad (11)$$

where $dr/db = -1/b^2$, $\partial Z/\partial r = W^r \ln W$ and $\partial Z/\partial W = rW^{r-1} = W^{\frac{1-b}{b}}/b$. Substituting these expressions and equation (10) into equation (11) produces:

$$\begin{aligned} \frac{dZ}{db} &= \frac{1}{b} W^{\frac{1-b}{b}} \frac{1}{N}\sum_{i=1}^{N} X_i^b \ln X_i - \frac{W^r \ln W}{b^2} \\ &= \frac{1}{b}\left(\frac{1}{N}\sum_{i=1}^{N} X_i^b\right)^{\frac{1-b}{b}} \frac{1}{N}\sum_{i=1}^{N} X_i^b \ln X_i - \frac{\left(\frac{1}{N}\sum_{i=1}^{N} X_i^b\right)^{\frac{1}{b}} \ln \frac{1}{N}\sum_{i=1}^{N} X_i^b}{b^2} \end{aligned} \qquad (12)$$

Simplifying gives:

$$\frac{dZ}{db} = \frac{1}{b^2}\frac{\left(\frac{1}{N}\sum_{i=1}^{N} X_i^b\right)^{\frac{1}{b}}}{\frac{1}{N}\sum_{i=1}^{N} X_i^b}\left(b\frac{1}{N}\sum_{i=1}^{N} X_i^b \ln X_i - \frac{1}{N}\sum_{i=1}^{N} X_i^b \ln \frac{1}{N}\sum_{i=1}^{N} X_i^b\right)$$

$$= \frac{1}{b^2}\frac{\left(\frac{1}{N}\sum_{i=1}^{N} X_i^b\right)^{\frac{1}{b}}}{\frac{1}{N}\sum_{i=1}^{N} X_i^b}\left(\frac{1}{N}\sum_{i=1}^{N} X_i^b \ln X_i^b - \frac{1}{N}\sum_{i=1}^{N} X_i^b \ln \frac{1}{N}\sum_{i=1}^{N} X_i^b\right) \qquad (13)$$

Provided that $X_i \geq 0$ for all *i*, and $X_i > 0$ for some *i* (as we can expect when the X_i represent the numerical opinions or views of different people), then

$$\frac{1}{b^2}\frac{\left(\frac{1}{N}\sum_{i=1}^{N} X_i^b\right)^{\frac{1}{b}}}{\frac{1}{N}\sum_{i=1}^{N} X_i^b} > 0 \qquad \text{for all } b \qquad (14)$$

Hence the sign of dZ/db will be the same as the sign of *D*, given by:

$$D = \frac{1}{N}\sum_{i=1}^{N} X_i^b \ln X_i^b - \frac{1}{N}\sum_{i=1}^{N} X_i^b \ln \frac{1}{N}\sum_{i=1}^{N} X_i^b$$
$$= \overline{X^b \ln X^b} - \overline{X^b} \ln\left(\overline{X^b}\right) \qquad (15)$$

Now putting

$$g(Y) = Y \ln Y \qquad (16)$$

allows us to re-express the components of the second line of equation (15) as

$$\overline{g(X^b)} = \overline{X^b \ln X^b} \qquad (17)$$

and

$$g\left(\overline{X^b}\right) = \overline{X^b} \ln \overline{X^b} \qquad (18)$$

Hence equation (15) may be written:

$$D = \overline{g(X^b)} - g\left(\overline{X^b}\right) \qquad (19)$$

The function, $g(y) = y \ln y$, has a minimum at $y = 1/e$ (since $dg/dy = 0$ when $\ln y = -1$) and is convex, as shown in Figure 1. Hence $D > 0$ by Jensen's Inequality (Jensen, 1906, Chiang, 1980). Therefore the derivative, dZ/db, must be positive for all *b*:

$$\frac{dZ}{db} > 0 \qquad \text{for all } b \qquad (20)$$

implying that *Z* is a strictly increasing function of the power, *b*. Moreover, since the back-transformed mean from the logarithmic transformation yields the limiting value of *Z* for the power transformation when $b \to 0$, from either below or above, as shown in Appendix A, it follows that the derivative, dZ/db, will obey equation (20) for the logarithmic transformation as well as for the straightforward power transformations.

Take in Figure 1.

Hence for any two powers, b_1 and b_2, leading to back-transformed means, Z_1 and Z_2 respectively, then $b_2 > b_1$ implies $Z_2 > Z_1$. Thus the arithmetic mean, where $b = 1$, will be greater than the geometric mean, where $b \to 0$. Similarly the r.m.s. value, where $b = 2$, will be greater than the arithmetic mean. And so on.

2.2 Example

As a numerical demonstration of these findings, a simulation was made of an opinion poll in which people would be asked to place a monetary value on the average total of life-years to come for a UK citizen – the "value of a human life". A random sample of 100 was generated with the distribution of values shown in Figure 2, which were in a range up to £4M.

Take in Figure 2.

Back-transformed means were found for power transformations in the range $-1 \le b \le 2$, where $b = 0$ corresponds to the logarithmic transformation and $b = 1$ corresponds to the sample mean. Figure 3 corroborates the result proved in Section 2.1, namely that the higher the value of the power *b*, the higher the back-transformed mean, *Z*. This result carries over unchanged to negative powers, *b*, as well as positive powers.

Take in Figure 3.

Clearly transformations with a power, *b*, less than unity will give a back-transformed mean that is below the sample mean, while transformations with a power greater than unity will yield back-transformed means above the sample mean, and this includes the important case of the geometric mean, the limiting form of the power transformation as $b \to 0$. As Cauchy predicted, the geometric mean is less than the sample mean, with the extent of the shortfall depending on the distribution. For the distribution shown in Figure 2, the geometric mean turns out to be £1.047 M, whereas the sample mean is 64% greater at £1.714 M.

This difference would be significant for policy making, and the question arises: which mean should be adopted? Already we can see the problem and its importance. If the person analysing the opinion poll data is given a free choice of transformation, then should he want a high figure (in this instance perhaps he might be contracted to a manufacturer selling protection equipment) then he might be drawn to a high transformation power, e.g. $b = 2$ and hence the r.m.s. value that will provide him with the apparently justifiable large figure of £2.123 M. On the other hand if he were to want a low figure for some reason (such as a desire to save money on safety interventions in this case), he might be drawn to transformation powers, b, that will give him a small figure, e.g. $b = 0$ and the geometric mean, which is only half the previous figure.

The question of how to ensure objectivity in the chosen mean will be addressed fully later in the paper, but first a new result will be given that extends that proved in Section 2.1 for power transformations. It concerns the relative size of the back-transformed mean and the sample mean for strictly increasing functions whose nonlinearity is of a more general nature.

3 The size of the back-transformed mean relative to the sample mean for general, nonlinear, strictly increasing transformations

This section will prove that general transformations that are convex will yield a back-transformed mean that is greater than the sample mean. (Figure 1 gives an example of a convex function, which is strictly increasing for $y > 1/e$.) Meanwhile general, concave transformations will give rise to a back-transformed mean that is always less than the sample mean. (Figure 4 gives an example of a concave function.) Linear transformations will yield a back-transformed mean that is the same as the sample mean.

By way of diversity, an example will be given of a concave transformation that is not a member of the family of power transformations. Here the new theory enables deductions to be made about this transformation that would be difficult to reach via another route.

If the analyst decides to employ a general, nonlinear transformation but wants to demonstrate his objectivity, he is left with a similar problem to that he incurs when using a power transformation. For in this case, too, he can decide in advance the size of the back-transformed mean of his choosing relative to that of the sample mean.

3.1 Analysis

Let the general, nonlinear, strictly increasing transformation be a function of the i^{th} view:

$$Y_i = g(X_i) \qquad (21)$$

where the term, "nonlinear", is held to include linear as a special case. The average of the transformed data is given by

$$W = \frac{1}{N}\sum_{i=1}^{N} Y_i \qquad (22)$$

The back-transformed mean, *Z,* will now obey:

$$\begin{aligned} g(Z) &= W \\ &= \frac{1}{N}\sum_{i=1}^{N} g(X_i) \qquad (23) \\ &= \overline{g(X)} \end{aligned}$$

We shall be discussing the relative size of the back-transformed mean and the sample mean, and, for enhanced clarity, we shall give the sample mean its unique symbol, *V*, where $V = \overline{X} = \sum_{i=1}^{N} X_i / N$, so that $g(V) = g(\overline{X})$. The relative sizes of $g(Z) = \overline{g(X)}$ and $g(V) = g(\overline{X})$ now follow from Jensen's inequality (Jensen, 1906). We may distinguish three cases.

If $g(.)$ is strictly concave, then $\overline{g(X)} < g(\overline{X})$, so that $g(Z) < g(V)$. Since $g(.)$ is strictly increasing, it follows that $Z < V$: the back-transformed mean is less than the sample mean.

Conversely, if $g(.)$ is strictly convex, then $\overline{g(X)} > g(\overline{X})$, so that $g(Z) > g(V)$. Since $g(.)$ is strictly increasing, then $Z > V$.

Finally, if $g(.)$ is linear, the two expressions will be the same, so that $Z = V$.

These results accord with the results derived in Section 2 for power transformations. Power transformations with $0 \le b < 1$ (including ln *x* as $b \to 0$) are strictly increasing and concave, so that $Z < V$, as shown previously. Considering power transformations with $b > 1$, such as $b = 2$, which back-transforms to the r.m.s. value, these are strictly increasing and convex, so that $Z > V$. Meanwhile any positive linear transform, $b = 1$, will back-transform to the sample mean: $Z = V$.

Power transformations with the structure, $b^{-1}(X^b - 1)$, are strictly increasing and concave for $b < 0$, from which it follows that their back-transformed mean must be less than the sample mean: $Z < V$. Since, moreover, the back-transformed mean for the transformation, X^b, is the same as that for $b^{-1}(X^b - 1)$ for all *b*, it follows that the main result of this section can be extended to power transformations of the form, $X^b; b < 0$. For these transformations, too, $Z < V$. Thus a further result from Section 2 is corroborated.

3.2 Example

As a example of a nonlinear function diverse from a power transformation, consider the arc tan transformation, whereby all numbers, X, on the non-negative real axis are regarded as tangents of some angle between 0 and $\pi/2$: $X = \tan(\alpha)$. Each notional angle may be found by the transformation:

$$\alpha_i = \arctan(X_i) \qquad (24)$$

and an average, W, taken:

$$W = \frac{1}{N}\sum_{i=1}^{N}\alpha_i \qquad (25)$$

The back-transformed mean is then:

$$Z = \tan(W) \qquad (26)$$

When $N = 2$, Z is the tangent of the average of two angles and may also be regarded as embodied in the vector $(Z,1)$ that bisects the angle between the vectors $(X_1,1)$ and $(X_2,1)$. An analytical solution may be found as (see Appendix B):

$$Z = \frac{X_2\sqrt{1+X_1^2} + X_1\sqrt{1+X_2^2} - X_1 - X_2}{X_1X_2 + \sqrt{1+X_1^2} + \sqrt{1+X_2^2} - \sqrt{(1+X_1^2)(1+X_2^2)} - 1} \qquad \text{(B.5)}$$

The complicated form of equation (B.5) means that is not easy to determine what are the relative sizes of the back-transformed mean, Z, and the sample mean, V, under all circumstances:

$$V = \frac{X_1 + X_2}{2} \qquad (27)$$

However, the arc tan transformation of equation (24) is strictly increasing and strictly concave, as shown in Figure 4. Hence the result derived in Section 3.1 will apply, and we know that the back-transformed mean, Z, will always be less than the sample mean, V, for the non-degenerate case where the sample contains more than one measurement, not all identical.

<u>Take in Figure 4.</u>

We may conclude from Section 3 that it is possible in advance and at will to select a nonlinear function, $g(.)$, that will give a back-transformed mean that is higher than, lower than or the same as the sample mean. This raises a similar question on objectivity to the one that arose in Section 2: if the analyst can chose his mean to be either larger than or less than the sample mean, how can we be assured of the objectivity of his result?

4. Structural view independence

This section shows that the views of the participants in a survey can be regarded as evolving over time, which implies that the consolidated view will also develop over time. This development process may be monitored when focus groups are used, for example, and may be modelled using the techniques of dynamic simulation.

By this dynamic model, the final consolidated view, *Z*, may be regarded as dependent on an evolution of the participants' views, with that dependency quantified by a set of sensitivity functions, $\partial Z/\partial X_i$, with respect to each person's view, X_i. These sensitivity functions are used as the basis for the criterion of structural view independence, which is introduced and defined in this section. Structural view independence is an implication of the equal rights philosophy and requires that the consolidation algorithm should contain no structural bias that would render the views of some people more important than the views of others.

4.1 Analysis

In a focus group, a random sample of members of the public may be asked to consider an issue over a period of time, perhaps a day or an evening. Each person in the sample will have a starting view and a closing view that are potentially different, reflecting the fact that his or her thinking may develop with time; both views may be regarded as random variables.

The effect of each person's changed view on the back-transformed mean may be calculated using the method of referred derivatives (Thomas, 1997, 1999), which, although it may be understood most easily as simulating the effect of changes that occur smoothly over time, preserves the validity of its endpoint whether those changes are smooth or not.

Using the method of referred derivatives, the back-transformed mean, *Z,* will change with time according to:

$$\frac{dZ}{dt} = \mathbf{bu} \qquad (28)$$

where **b** is the Jacobian, 1 x *N* row vector of the sensitivity functions, $\partial Z/\partial X_i$:

$$\mathbf{b} = \begin{bmatrix} \frac{\partial Z}{\partial X_1} & \frac{\partial Z}{\partial X_2} & \cdots & \frac{\partial Z}{\partial X_N} \end{bmatrix} \qquad (29)$$

while **u** is the column vector of the time differentials:

$$\mathbf{u} = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_N \end{bmatrix} = \frac{d\mathbf{x}}{dt} \qquad (30)$$

where $\mathbf{x}$ is the N x 1 column vector of views:

$$\mathbf{x} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \qquad (31)$$

For a given set of time differentials, $\mathbf{u}$, the back-transformed mean at any time, t, may be found by integration from a starting value, $Z_0 = Z|_{\mathbf{x}=\mathbf{x}_0}$, computed directly from a starting set of views, $\mathbf{x}_0$. Hence

$$Z(t) = Z_0 + \int_{\tau=0}^{t} \mathbf{bu}\, d\tau \qquad (32)$$

If the final views are $\mathbf{x}_f = \begin{bmatrix} X_{f1} & X_{f2} & \ldots & X_{fN} \end{bmatrix}^T$ then $Z_f = Z|_{\mathbf{x}=\mathbf{x}_f}$ will be equal to $Z(t_f)$ if equation (32) is integrated to final time, t_f, for a vector of differentials, $\mathbf{u}$, each component of which obeys the simple law:

$$U_i = \frac{dX_i}{dt} = \frac{X_{fi} - X_{0i}}{t_f} \qquad (33)$$

It should be noted that if only the back-transformed mean of the final views, $\mathbf{x}_f$, is sought, $Z_f = Z|_{\mathbf{x}=\mathbf{x}_f}$, then the choice of the starting views, $\mathbf{x}_0$, may be arbitrary.

Using notions of system dynamics (Forrester, 1961, Sterman, 2000), the evolution of a view may be modelled more realistically as a first-order, exponential lag, so that

$$U_i = \frac{dX_i}{dt} = \frac{X_{fi} - X_{0i}}{\tau_c} \qquad (34)$$

where τ_c is a time constant. The integration is now allowed to proceed to a steady state, theoretically over an indefinite period and practically to at least five times the time constant. Equation (32), when taken together with a more realistic model of opinion forming, such as that given in equation (34), demonstrates and defines the causal link between the sensitivity functions, $\partial Z/\partial X_i$, and the back-transformed mean, Z.

Under the equal rights philosophy, we should expect an unbiased, consolidated statistic, Z, to be affected by any person changing his or her view. Z would not respond to person i changing his view if the sensitivity function, $\partial Z/\partial X_i$, were zero for person, i, and so the statistic Z will not be fair if $\partial Z/\partial X_i = 0$ for some i. A dependence of $\partial Z/\partial X_i$ on one or more of the views, X_i, would suggest that the change in *Z* brought about by person *i*'s change in view would be influenced by the starting size of that view or of another view or of several views. To explore these ideas further, we propose the following definition.

Definition: we define a consolidated statistic, *Z*, to possess **structurally view independence** if and only if all of its sensitivity functions, $\partial Z/\partial X_i, i = 1,2,...,N$, are non-zero and independent of any of the views, $X_i, i = 1,2,...,N$.

In general, for any given transformation and sample size, *N*, the back-transformed mean, *Z,* can depend only on a set of constants and the *N* views in the sample, **x**, and so may be described fully by:

$$Z = Z(\mathbf{x}) \qquad (35)$$

It follows that the partial derivative, $\partial Z/\partial X_i$, can depend only on these views or a subset of them:

$$\frac{\partial Z}{\partial X_i} = \frac{\partial Z}{\partial X_i}(\mathbf{x}) \text{ for } i = 1,2,...,N \qquad (36)$$

Hence if

$$\frac{\partial Z}{\partial X_i} \neq \frac{\partial Z}{\partial X_j} \quad \text{for any } i, j \qquad (37)$$

the difference can have been caused only by the views, **x**, having an effect on $\partial Z/\partial X_j$ that is different from their effect on $\partial Z/\partial X_i$. But in the case of structural view independence, the views should have an effect on neither sensitivity function, implying that inequality (37) cannot hold. Therefore when the back-transformed mean is structurally view independent,

$$\frac{\partial Z}{\partial X_i} = \frac{\partial Z}{\partial X_j} \quad \text{for all } i, j \qquad (38)$$

None of the partial differentials, $\partial Z/\partial X_i, i = 1,2,...,N$, can depend on **x**, in the case of structural view independence. Hence each must be a constant, and, by equation (38), the same constant. This constant cannot be zero, however, since this would imply that all $\partial Z/\partial X_i, i = 1,2,...,N$ were zero, which is invalid under the definition of structural view independence. It would also imply that the back-transformed mean would be completely independent of all the views in the sample, which would render the process of taking views nonsensical.

5. The sensitivity functions for a general, nonlinear transformation under the condition of structural view independence

This section applies the concept of structural view independence to a general, nonlinear transformation, subject only to the condition that it is differentiable. The structural view independence condition of equation (38), that the sensitivity function should be the same for all views: $\partial Z/\partial X_i = \partial Z/\partial X_j$ for all i and j, is strengthened to the more specific: $\partial Z/\partial X_i = 1/N$ for all i. It is shown that, of all the power transformations, only the linear transformation can satisfy this condition for structural view independence, when the back-transformed mean becomes identical to the sample mean.

5.1 Analysis

The values of $\partial Z/\partial X_i, i = 1,2,...,N$ will now be explored for a general, nonlinear transformation, as described by equations (21) to (23) of Section 3, but without the constraint that the transformation be strictly increasing. Assuming $g(.)$ to be differentiable, we may differentiate equation (23) with respect to Z and invert the result to give

$$\frac{dZ}{dW} = 1 \Bigg/ \frac{dg(Z)}{dZ} \qquad (39)$$

Meanwhile, differentiating equation (22) with respect to Y_i gives

$$\frac{\partial W}{\partial Y_i} = \frac{1}{N} \qquad (40)$$

Differentiating equation (21) with respect to X_i gives

$$\frac{dY_i}{dX_i} = \frac{dg(X_i)}{dX_i} \qquad (41)$$

Hence the sensitivity function will be:

$$\begin{aligned} \frac{\partial Z}{\partial X_i} &= \frac{dZ}{dW}\frac{\partial W}{\partial Y_i}\frac{dY_i}{dX_i} \\ &= \frac{\dfrac{dg(X_i)}{dX_i}}{\dfrac{dg(Z)}{dZ}}\frac{1}{N} \end{aligned} \qquad (42)$$

or, after defining $h(x) = dg(x)/dx$

$$\frac{\partial Z}{\partial X_i} = \frac{h(X_i)}{h(Z)}\frac{1}{N} \qquad (43)$$

By equation (38), structural view independence imposes the condition that

$$\frac{h(X_i)}{h(Z)} = k \qquad \text{for all } i \qquad (44)$$

where *k* is some dimensionless constant. But $h(X_i)/h(Z)$ will be a random variable and not a constant if either or both the following conditions pertain:

(i) $h(X_i)$ has a non-null dependence on X_i or
(ii) $h(Z)$ has a non-null dependence on Z.

Since a random variable will not have a constant value, we may conclude first that, for equation (44) to hold, $h(X_i)$ must be independent of the value of X_i. Given that X_i may take any non-zero value, it follows that $h(X_i) = c$ for $0 \le X_i \le \infty$, where *c* is a constant. This range includes the range of possible *Z*, and so we may conclude that $h(Z) = c$ also. Hence

$$\frac{h(X_i)}{h(Z)} = \frac{c}{c} = 1 \qquad (45)$$

Substituting from equation (45) into equation (43) gives the condition for structural view independence for a general, nonlinear transformation, subject only to the condition that it is differentiable:

$$\frac{\partial Z}{\partial X_i} = \frac{1}{N} \qquad \text{for all } i \qquad (46)$$

Equation (46) quantifies the finding of equation (38) that, under structural view independence, the sensitivity of the back-transformed mean is the same for all participants in the survey.

The sensitivity function given by equation (46) may be identified as that associated only with the linear transformation out of all the power transformations:

$$Y = aX + c \qquad (47)$$

which leads to a back-transformed mean, *Z*, that is the sample mean:

$$Z = \frac{1}{N}\sum_{i=1}^{N} X_i \qquad (48)$$

(see equations (3) to (6), with $b = 1$). It will be shown in Section 6.2 that it is only the linear transformation out of all the general, nonlinear, increasing transformations that exhibits structural view independence.

6. Properties of structural view dependence

This section examines the properties of a general transformation exhibiting structural view dependence. Section 6.1 uses dimensional analysis (Buckingham Pi) to show that when a transformation exhibits structural view dependence, at least one sensitivity function will show a dependence on two or more views. Thus in structural view dependence, at least one person's view is "edited" by reference to the view or views of at least one other person in the sample.

More precisely, the dependency will be on ratios of views. Hence under structural view dependence the sensitivity function for at least one person will depend on the size of one or more views relative to the size of the view of someone else in the sample. An alternative interpretation afforded by dimensional analysis is that the sensitivity function will depend on the size of one or more views relative to the back-transformed mean. In this case, view of the person concerned is "edited" by reference to those of everyone else in the sample. The degree of editing will then vary depending on the size of the view.

Section 6.2 shows that the condition for structural view independence, $\partial Z / \partial X_i = 1/N$ for all i, is violated for all nonlinear transformations, both concave and convex, except for the limiting case of the linear transformation, when the back-transformed mean is equal to the sample mean. Thus the only mean that satisfies the requirement for structural view independence is the sample mean.

6.1 General form of the sensitivity function

If the back-transformed mean is structurally view dependent, then, by definition, one or more of the sensitivity functions will depend on a number, *n*, of the views, where $1 \le n \le N$:

$$\frac{\partial Z}{\partial X_i} = f_i(X_1, X_2, ..., X_n) \qquad \text{for some } i \qquad (49)$$

An important case occurs where each view has units of currency, when dimensional analysis (Buckingham, 1914, Sonin, 2001, 2004) may be applied to draw general conclusions. The *n* views defining $\partial Z / \partial X_i$, namely X_k, $k = 1,2,...,n$, form a set that may be regarded as both complete, in the sense that no other variable can affect the sensitivity function, and independent, in that the value of each view can be altered arbitrarily without affecting any other. We may non-dimensionalise equation (49) by first noting that each of the independent variables has the dimension of currency, *C*, where *C* may be £ or $ or €, for example:

$$[X_k] = C \qquad \text{for } k = 1,2,...,n \qquad (50)$$

where the square brackets indicate the operation of finding the dimension.

Meanwhile the dependent variable is the sensitivity function, which must be dimensionless:

$$\left[\frac{\partial Z}{\partial X_i}\right]=1 \qquad \text{for all } i \qquad (51)$$

since the back-transformed mean, Z, and the views, $X_k, k=1,2,...,i,...,N$, share the same dimension of currency.

A subset containing just one member, X_K, may be selected from the n independent variables listed in equation (49), $X_k, k=1,2,...,K,...,n$. This subset will be complete in the sense that the dimensions of the remaining members of the set, $X_k, k=1,2,...,K-1,K+1,...,n$, can be expressed in terms of the dimensions of the subset – each has the same dimension for the important case where the views are denominated in currency, of course. Moreover, the subset containing X_K is dimensionally independent, since none of its members has a dimension that can be expressed in terms of the dimensions of the remaining members – the fact that there is only one member in the subset makes this is a trivial result, since there are no remaining members once a selection has been made. Hence the subset fulfils the requirement of the Buckingham Pi procedure, and we may use its single member, X_K, to non-dimensionalise each of the other independent variables by division:

$$\Pi_k=\frac{X_k}{X_K} \qquad k=1,2,...,K-1,K+1,...,n \qquad (52)$$

Meanwhile, since $\partial Z/\partial X_i$ is already dimensionless, we may set $\Pi_0=\partial Z/\partial X_i$ and use the canonical Buckingham Pi equation:

$$\Pi_0=q_i\left(\Pi_1,\Pi_2,...,\Pi_{K-1},\Pi_{K+1},...,\Pi_n\right) \qquad (53)$$

where $q_i(.)$ is some function to give

$$\frac{\partial Z}{\partial X_i}=q_i\left(\frac{X_1}{X_K},\frac{X_2}{X_K},...,\frac{X_{K-1}}{X_K},\frac{X_{K+1}}{X_K},...,\frac{X_n}{X_K}\right) \qquad (54)$$

By equation (54), the number of independent variables has been reduced from n to $n-1$. Since at least one independent variable must remain for structural view dependence to be exhibited in the sensitivity function under consideration, it follows that the condition on n for structural view dependence is $n \geq 2$, and, since $N \geq n$, it follows that $N \geq 2$. This demonstrates that structural view dependence can occur as soon as the sample contains 2 or more views. An example illustrating this fact for the geometric mean will be given in Section 7.

An important result from equation (54) is that the sensitivity function under structural view dependence will depend on ratios of views – the size of one view relative to another. The selection of the single member, X_K, of the subset is arbitrary, and so the view may be measured relative to any other view. Hence under structural view dependence the sensitivity function for at least one person will depend on the size of one or more views relative to the size of the view of someone else in the sample.

As an alternative, we might choose to rewrite equation (49) in the fuller form:

$$\frac{\partial Z}{\partial X_i} = f_i(X_1, X_2, ..., X_N) \qquad \text{for some } i \qquad (55)$$

where the function, $f_i(.)$, is now assumed to include the appropriate null-dependencies for those $N-n$ views having no effect on $\partial Z/\partial X_i$. Equation (55) may be recast in terms of the back-transformed mean, *Z*, as:

$$\frac{\partial Z}{\partial X_i} = \phi_i(X_1, X_2, ..., X_i, ..., X_{N-1}, Z) \qquad \text{for some } i \qquad (56)$$

since $Z = Z(X_1, X_2, ..., X_N)$. Selecting *Z* as the single member of the subset and following a similar procedure of dimensional analysis to that followed above allows equation (54) to be rewritten as:

$$\frac{\partial Z}{\partial X_i} = \psi_i\left(\frac{X_1}{Z}, \frac{X_2}{Z}, ..., \frac{X_{N-1}}{Z}\right) \qquad (57)$$

Equation (57) shows the general form that any sensitivity function exhibiting structural view dependence will have. Under this interpretation, the sensitivity function will depend on the size of one or more views relative to the back-transformed mean. Examples of this form will be derived in Section 7 for power transformations.

6.2 Relative weightings

The relative weightings given to modifications to views of different sizes can be seen for a transformation, $g(x)$, whose the first derivative, $h(x) = dg/dx$, is non-negative but strictly decreasing in *x*. Concave transformations take this form, such as the power transformations with $0 < b < 1$. Now if $X_i > Z$, then

$$h(X_i) < h(Z) \qquad (58)$$

Hence from equation (43):

$$\frac{\partial z}{\partial X_i} < \frac{1}{N} \qquad \text{for } X_i > Z \qquad (59)$$

Conversely, if $X_i < Z$, then

$$h(X_i) > h(Z) \qquad (60)$$

in which case equation (43) implies

$$\frac{\partial Z}{\partial X_i} > \frac{1}{N} \qquad \text{for } X_i < Z \qquad (61)$$

Thus when the transformation is concave, more weight will be given to opinions that are low relative to the back-transformed mean, while those that are high will be downplayed.

The reverse applies for transformations, $g(x)$, whose the first derivative, $h(x) = dg/dx$, is non-negative but strictly increasing in *x*. Convex transformations are of this form, for example power transformations with $b > 1$. Now if $X_i > Z$, then

$$h(X_i) > h(Z) \qquad (62)$$

and so, from equation (43):

$$\frac{\partial Z}{\partial X_i} > \frac{1}{N} \qquad \text{for } X_i > Z \qquad (63)$$

On the other hand, if $X_i < Z$, then

$$h(X_i) < h(Z) \qquad (64)$$

so that:

$$\frac{\partial Z}{\partial X_i} < \frac{1}{N} \qquad \text{for } X_i < Z \qquad (65)$$

This shows that when the transformation is convex, more weight will be given to opinions that are high relative to the back-transformed mean, while those that are low in comparison will be downplayed.

Conditions (59) and (61) violate the necessary condition for structural view independence given by equation (46), showing that general, nonlinear, differentiable transformations that are concave cannot be structurally view independent. Meanwhile, conditions (63) and (65) violate the necessary condition for structural view independence of equation (46), demonstrating that general, nonlinear, differentiable transformations that are convex cannot be structurally view independent. Hence the condition for a general, nonlinear, differentiable transformation to possess structural view independence is that it must be the limiting case of a linear transformation.

It may be concluded that of the broad category of transformation just discussed, only a linear transformation will exhibit structural view independence, in which case the back-transformed mean will be equal to the sample mean. All views will then be treated equally, with each person's contribution weighted by the inverse of the number in the sample.

7. Structural view dependence and independence for power transformations

Power transformations are specific, defined instances of the general, nonlinear, increasing, differentiable transformations introduced in Section 3 and discussed further in Section 5 and Section 6.2. For example, power transformations with $0 \le b < 1$ are increasing and concave and, while power transformations with $b > 1$ are increasing and convex. Thus they constitute good vehicles for testing and illustrating the results derived in Section 6.

It will be shown first that the sensitivity function when structural view dependence occurs depends on the ratio of the view to the back-transformed mean, as claimed in equation (57). Next, the results of Section 6.2 will be corroborated: when the power transformation is convex ($b > 1$), then views higher than the back-transformed mean will be given greater emphasis and views lower than the back-transformed mean will be given lower emphasis. But if the transformation is concave ($b < 1$), then views lower than the back-transformed mean will be given greater emphasis and views higher than the back-transformed mean will be given lower emphasis. Only if the power transformation is linear will the emphasis be the same for all views.

Then a demonstration will be given, using the logarithmic transformation and hence geometric mean, that structural view dependence can occur as soon as there are two or more views in the sample.

A synthesis will be made at the end of the section, drawing together the facts that $b > 1$ implies that the back-transformed mean

i. will be more sensitive to high views, and
ii. will be greater than the sample mean,

while $b < 1$ implies that the back-transformed mean

i. will be more sensitive to low views, and
ii. will be less than the sample mean.

The results are pointed out to be two sides of the same coin. Hence it will have been demonstrated how an analyst, by choosing a power, *b*, lower than the value of unity associated with the linear transformation and sample mean, will have guaranteed that his back-transformed mean will be less than the sample mean, and that this result has been achieved by systematically downgrading the importance accorded to people with high views. The converse applies if the analyst chooses a high power, $b > 1$.

7.1 Analysis

From equations (8) and (9), the sensitivity function will take the form:

$$
\begin{aligned}
\frac{\partial Z}{\partial X_i} &= \frac{dZ}{dW}\frac{\partial W}{\partial X_i} = \frac{1}{b}W^{\frac{1}{b}-1}.\frac{1}{N}bX_i^{b-1} \\
&= W^{\frac{1-b}{b}}X_i^{b-1}\frac{1}{N} = (Z)^{1-b}X_i^{b-1}\frac{1}{N} \\
&= \left(\frac{X_i}{Z}\right)^{b-1}\frac{1}{N}
\end{aligned}
\qquad (66)
$$

It is clear from equation (66) that structural view independence will occur if and only if $b = 1$, when equation (66) becomes identical with equation (46). Structural view dependence will occur for all other values of *b*, and the form of equation (57) has been confirmed, whereby the sensitivity function when structural view dependence occurs depends on the ratio of the view to the back-transformed mean.

Suppose that view, X_i, is greater than the back-transformed mean, *Z*. If $b > 1$, then $(X_i/Z)^{b-1} > 1$ and thus the sensitivity obeys $\partial X_i/\partial Z > 1/N$. But if $b < 1$, then $(X_i/Z)^{b-1} < 1$ and thus the sensitivity conforms to $\partial X_i/\partial Z < 1/N$.

Suppose, on the other hand that view, X_i, is less than the back-transformed mean, *Z*. If $b > 1$, then $(X_i/Z)^{b-1} < 1$ and thus the sensitivity obeys $\partial X_i/\partial Z < 1/N$. But if $b < 1$, then $(X_i/Z)^{b-1} > 1$ and so the sensitivity function obeys $\partial X_i/\partial Z > 1/N$.

As noted previously, if $b = 1$ then $\partial X_i/\partial Z = 1/N$.

Summarising for power transformations, if the power, *b*, is greater than unity, then views higher than the back-transformed mean will be given greater emphasis and views lower than the back-transformed mean will be given lower emphasis. If the power, *b*, is less than unity, then views lower than the back-transformed mean will be given greater emphasis and views higher than the back-transformed mean will be given lower emphasis. Only if the power transformation is linear will the emphasis be the same for all views. These findings for power transformations corroborate the conclusions drawn in Section 6.2 for general convex and concave functions.

Equation (66) with *b* = 0 gives the sensitivity function associated with the logarithmic transformation, the back-transformed mean of which is equal to the geometric mean, which may be written:

$$
\begin{aligned}
Z &= e^{\frac{1}{N}(\ln X_1 + \ln X_2 + \ldots \ln X_i + \ldots \ln X_N)} = e^{\left(\ln X_1^{\frac{1}{N}} + \ln X_2^{\frac{1}{N}} + \ldots \ln X_i^{\frac{1}{N}} + \ldots \ln X_N^{\frac{1}{N}}\right)} \\
&= X_1^{\frac{1}{N}}X_2^{\frac{1}{N}}\ldots X_i^{\frac{1}{N}}\ldots X_N^{\frac{1}{N}} = \left(\prod_{i=1}^{N} X_i\right)^{\frac{1}{N}}
\end{aligned}
\qquad (67)
$$

The sensitivity function may be found by partial differentiation:

$$\frac{\partial Z}{\partial X_i} = \left(X_1^{\frac{1}{N}} X_2^{\frac{1}{N}} .. X_{i-1}^{\frac{1}{N}} X_{i+1}^{\frac{1}{N}} ... X_N^{\frac{1}{N}} \right) \frac{1}{N} X_i^{\frac{1}{N}-1}$$
$$= \frac{Z}{X_i} \frac{1}{N} \qquad (68)$$

confirming the result of (66) for $b = 0$. Equations (67) and (68) show how all N views are needed to determine the sensitivity function, $\partial Z/\partial X_i$, for the geometric mean, implying that $n = N$ in this case.

Let us consider the logarithmic transformation for the case where there are only 3 views in the sample: $N = 3$ and the 3 views are given the subscripts, i, j and k. From equations (67) and (68)

$$\frac{\partial Z}{\partial X_i} = \frac{1}{3} \left(\frac{X_j}{X_i} \frac{X_k}{X_i} \right)^{\frac{1}{3}} \qquad (69)$$

Equation (69) shows that, while $\partial Z/\partial X_i = \partial Z/\partial X_i \left(X_i, X_j, X_k \right)$, the sensitivity may be regarded also as a function of the 2 dimensionless groupings, X_j/X_i and X_k/X_i, in line with the dimensional analysis of Section 6. This dependency shows that structural view dependence occurs. When the number of views in the sample is reduced to $N = 2$, then the sensitivity function will take the form:

$$\frac{\partial Z}{\partial X_i} = \frac{1}{2} \left(\frac{X_j}{X_i} \right)^{\frac{1}{2}} \qquad (70)$$

Hence $\partial Z/\partial X_i = \partial Z/\partial X_i \left(X_i, X_j \right)$, but the sensitivity may be regarded also as dependent on the single dimensionless grouping, X_j/X_i. Structural view dependence has thus been demonstrated even when there are only 2 views in the sample. If, however, the sample size is reduced to $N = 1$, the back-transformed mean will be the same as the single view, $Z = X_i$. Hence from equation (68):

$$\frac{\partial Z}{\partial X_i} = 1 \qquad (71)$$

Now the sensitivity function will be independent of all views, implying that structural view dependence has disappeared for $n = N < 2$. This corroborates the dimensional analysis of structural view dependence given in Section 6.

This section has shown that it is possible to adjust the power, b, used in the power transformation so as to render the back-transformed mean either more sensitive to high views or more sensitive to low views, in advance and at will. This result parallels that of Section 2, where it was shown that it was possible to adjust the power, b, in advance and at will, to determine the size of the back-transformed mean relative to the sample mean. In fact, the results are two sides of the same coin. As explained in Section 4 and as will be illustrated numerically in Section 9, producing a back-transformed mean that is lower than the sample mean requires that the back-transformed mean be more sensitive to low views than high views. Conversely, a back-transformed mean higher than the sample mean is brought about only when the back-transformed mean is more sensitive to high views than low views.

Combining the two results, we may see that a value of *b* greater than unity will result in structural view dependence and lead to a back-transformed mean that is (i) greater than the sample mean and (ii) more sensitive to high views than low views. Meanwhile, a value of *b* less than unity will result in structural view dependence once more, but lead to a back-transformed mean that is (i) lower than the sample mean and (ii) more sensitive to low views than high views. The logarithmic transformation, producing the geometric mean as its back-transformed mean, has a value of zero for *b*, and so corresponds to the second case of a lower back-transformed mean.

By contrast, the sample mean gives equal treatment to all views, high or low.

8. Testing trimmed means and the median for structural view independence

Both repeated measurements of the same physical quantity and surveys of the views of many people can result in outliers – data points that are significantly different from the bulk of observations. The way in which they are treated needs to be different, however, if each person's view is to be accorded equal worth.

In the case of physical measurements, even where no clear reason can be established for one or more measurements being significantly different from the rest, but it is believed that a single value will pertain, a method of proceeding sometimes used in practice is to reject data points in pairs. The idea is that some degree of compensation for the censoring of an outlier is achieved by rejecting a data point at the other end of the spectrum. This is the process by which trimmed means are generated (Rice, 2007). The procedure for finding the double-sided, 100α % trimmed mean, Z_α, is to place the data in order of size, discard the lowest 100α % and the highest 100α %, and then take the arithmetic mean of the remaining data points.

If such a process were to be applied to views, it would be represented by:

$$\begin{aligned} Z_\alpha = {} & 0.X_1 + 0.X_2 + \ldots 0.X_{N\alpha} + \frac{1}{M}\left(X_{N\alpha+1} + X_{N\alpha+2} + \ldots + X_{N(1-\alpha)}\right) \\ & + 0.X_{N(1-\alpha)+1} + 0.X_{N(1-\alpha)+2} + \ldots + 0.X_{N-1} + 0.X_N \end{aligned} \qquad (72)$$

where the views, $X_i, i = 1,2,\ldots,N$, have now been put in size order and where $M = N(1-2\alpha)$ is the number of data points remaining after trimming. The subsample of size *M* is obviously not a random selection, as 2α % of the sample has been censored in a systematic way.

Although Z_α is a weighted mean rather than a back-transformed mean, it is possible to differentiate equation (72) with respect to X_i to produce its sensitivity function, $\partial Z_\alpha / \partial X_i$ as:

$$\frac{\partial Z_\alpha}{\partial X_i} = \begin{cases} 0 & \text{for } 1 \le i \le N\alpha \\ \dfrac{1}{M} & \text{for } N\alpha < i \le N(1-\alpha) \\ 0 & \text{for } N(1-\alpha) < i \le N \end{cases} \qquad (73)$$

It is thus clear that the sensitivity function, $\partial Z_\alpha / \partial X_i$ depends on the values of all the $X_i, i = 1,2,...,N$ and the procedure of finding the trimmed mean for human opinions must violate the conditions for structural view independence for any $\alpha > 0$. Hence trimming based solely on the size of the view recorded should not be used in the consolidation of views.

As pointed out by Rice (*ibid.*), the median may be regarded as the double-sided, 50% trimmed mean. Hence the procedure of finding the median for human opinions will violate the conditions for structural view independence. The median will be sensitive only to one opinion in an odd-numbered sample and two opinions in an even-numbered sample. Hence the median violates the conditions for structural view independence and should not be applied to the consolidation of human views into a single figure.

A trimmed mean is used in calculation of the London Inter-Bank Offered Rate (LIBOR), which is, in fact, a composite of rates compiled in London but relating to many different currencies and different lending periods. The LIBOR rates constitute global benchmarks, referenced in transactions with a notional outstanding value of at least \$300tn (Wheatley, 2012a,b). It is relevant to this paper that the LIBOR rate was found to be subject to manipulation over a period of many years, with Barclays Bank plc being fined £59.5m by the Financial Services Authority (FSA) on 27 June 2012. Barclays was fined \$360m separately by the US authorities at about the same time for attempted manipulation of and false reporting concerning LIBOR and the European Inter-Bank Offered Rate (EURIBOR) (Wheatley, 2012a). Further investigations are ongoing.

Wheatley (*op. cit.*) identified the major weakness that "submissions to LIBOR have become increasingly reliant on expert judgement rather than transaction data" but had to accept in his final report that the "transaction-based model is not a viable option in the short-term" because of the small number of transactions (Wheatley, 2012a,b). He concluded that an opinion survey needed to be retained to generate LIBOR rates, but recommended a set of strengthened administrative rules. It was stipulated that the LIBOR views supplied by the banks should be evidence-based wherever possible, with the submitting bank keeping records of the transactions that formed part of the bank's evidence base.

Wheatley rejected the option of taking the median of the views to exclude outliers, the ultimate trimming measure, as noted above. The previous practice of discarding half the sample through of double-sided 25% trimming was retained, however, even though such trimming was shown to provide no protection against the previous manipulation of LIBOR rates. In view of the very small remaining size of the sample – between 4 and 10 (British Bankers' Association, 2013) – it may well have been an unintended contributor to the manipulation, as illustrated by the following comments from "Manager D" of Barclays Bank in a routine liquidity telephone call to the FSA in March 2008. *The Daily Telegraph* (2013) reported him as saying that Barclays had been "picked upon for posting LIBORs above everyone else" in 2007 and he went on to ask, "what is everybody, open brackets to be honest, including ourselves close brackets, going to do?". His answer to his own, rhetorical question was, "Keep their heads below the parapet and not stick out". A month later in a similar telephone call, the same Manager D said "we did stick our heads above the parapet last year, got it shot off, and put it back down again. So, to the extent that, um, the LIBORs have been understated, are we guilty of being part of the pack? You could say we are. We've always been at the top end and therefore one of the four banks that's been eliminated. Um, so I would, I would sort of express us maybe as not clean clean, but clean in principle".

The use of an opinion surveys for determining LIBOR rates is highly problematic when each respondent is known to have an incentive to falsify his view, rendering all the views in the survey suspect. Normally one would hope to see all such potential respondents excluded at the outset, as discussed in the Introduction to this paper. It is not surprising, therefore, that Wheatley should comment in the Foreword to his final report: "So while this is the Wheatley Review's final report, I am well aware that it is far from the last word on the subject". Even when backed by an evidence base, the fact that there is a continuing requirement for judgement from a banking expert with an incentive to give a biased view does not provide comfort. The known motivation of all respondents to manipulate their responses takes the gathering and consolidation of views on LIBOR away from the normal run of opinion surveys and into the realms of game theory (von Neumann and Morgenstern, 1944, 2007). This is surely an area where revealed preferences must be favoured over stated preferences and research to find the fully objective approach currently lacking should be given high priority.

9. Numerical example illustrating structural view dependence

The example follows the development over a working day of the views of a sample of people, assumed to comprise a focus group tasked with deciding what an environmental protection system is worth to the public. The focus group is assumed to consist of just 7 people, which allows the graphs of the data generated to retain comprehensibility.

The data analyst is assumed, for whatever reason, to have decided to consolidate the 7 views into a geometric mean. The evolution of the geometric mean, the median and the mean will be shown. The sensitivity functions, $\partial Z/\partial X_i$, are calculated, and it will be shown that the view of the person who sets the least store by the scheme comes to dominate; not because he is the most knowledgeable or expert, but simply because his view is the lowest. The logarithmic transformation is thus demonstrated to possess structural view dependence and the geometric mean is shown to fail comprehensively the requirement for equal rights.

9.1 Example

A focus group comprising 7 people has been asked to spend a working day providing a valuation of the worth of a environmental protection scheme based on their own judgements. Each person sets out with the same view – a rare situation in reality, but useful as an initial condition simply for the purpose of illustration. Their common starting opinion is that the scheme is worth \$4 million, but as they mull the question over, their views begin to diverge. Individuals 1 to 6 move towards final opinions of \$0.5M, \$2M, \$3M, \$5M, \$6M and \$7.5M. The evolution of these opinions takes the form of an exponential lag with a time constant of 80 minutes in each case, so that the opinions will have reached a steady state by the end of the day, 8 hours later. By contrast, Person 7 acts analogously to a "floating voter" and hesitates in his view, which settles finally into a steady oscillation, with a peak of \$4.9M, a trough of \$3.1M and a period of 2 hours and 40 minutes.

It is assumed that the analyst wishes to apply the logarithmic transformation ($b \to 0$ in the power transformation) to the views in the sample, so that the back-transformed mean is the geometric mean.

Figure 5 shows the evolution of the views of the seven people over the day. The figure shows also the geometric mean and the sample mean, with the former falling below the latter from the start, as predicted by the theory of Section 2. The oscillation in the views of Person 7 induces a small oscillation in the geometric mean, which ends at \$3.1M +/- \$0.1M. The sample mean retains a central value of \$4M, with an end-of-day oscillation of +/- \$0.12M. There are always 3 people with views below that of Person 7 and 3 people with views above. Hence Individual 7's view is the median at all times.

<u>Take in Figure 5.</u>

Figure 6 shows the sensitivity functions for the 7 people. The sensitivity of the back-transformed mean to each view is seen to depend strongly on the size of the various views. Thus the oscillations in Person 7's view are clearly visible in the 3 highest sensitivity functions, corresponding to the 3 lowest views. The effect is still present but smaller in the 3 lowest sensitivity functions (corresponding to the 3 highest views).

<u>Take in Figure 6.</u>

All sensitivities are the same at the beginning when everyone holds the same view, but they soon diverge. The sensitivities to the 3 highest views steady out at below 0.1, while the sensitivities to the 3 lowest views are much higher. Indeed the sensitivity to the lowest view steadies out an order of magnitude higher than that of the highest views, reaching an average value of just under 0.9. It is, of course, these differences in sensitivity that account for the fall of the geometric mean well below the sample mean. It is clear that the higher views are discounted heavily in favour of the lower views, with the lowest being accorded very great influence.

If the median were chosen as nominally representative of the sample, then, since the median and the view of Person 7 are always the same, it follows that the sensitivity of the median to the views of the various sample members would be unity for Person 7, but zero for everyone else. This weighting, dependent entirely on the relative sizes of people's views, violates the concept of structural view independence.

10. Conclusions

The paper has examined the consolidation of the views of people in a sample into a single, representative value, where each view is a person's judgement of the size of a numerical parameter. The results presented are independent of the distributions of views in the population and in the sample.

The back-transformed mean arising from a power transformation has been considered first, because of the prevalent use of such transformations in statistical analysis. In a generalisation of Cauchy's result for the geometric mean, it has been shown that the back-transformed mean from a power transformation, including negative powers, is strictly increasing in the power, *b*. Hence, for example, choosing $b < 1$ means that the back-transformed mean, *Z*, will be less than the sample mean, produced when $b = 1$.

The theory is then extended to the case of general, nonlinear, increasing transformations, where a distinction is drawn between concave transformations, whose back-transformed mean is always less than the sample mean, and convex transformations, whose back-transformed mean is always greater than the sample mean. The two theoretical developments are linked by the fact that a power transformation with a power less than unity will be concave, while those with a power greater than unity will be convex, so that the power transformations are subsumed within the class of general, nonlinear, increasing transformations. Of the power transformations, only the linear will produce a back-transformed mean equal to the sample mean.

The concept of people's views developing over time has been introduced, by which the final consolidated view, *Z*, may be regarded as depending on an evolution of the participants' views, with that dependency quantified via a set of sensitivity functions, $\partial Z/\partial X_i$, with respect to each person's view, X_i. These sensitivity functions form the basis for the new criterion of structural view independence, which requires that the consolidation algorithm should contain no structural bias that would render the views of some people more important than the views of others. Such a situation would violate the equal rights approach that is needed when evaluating the views of people not debarred from offering an opinion for reasons of insanity, lack of knowledge, insufficient experience and so on.

It is shown that, for a transformation to possess structural view independence, all its sensitivity functions, $\partial Z/\partial X_i$, must be equal to the equal weighting figure, $1/N$, where *N* is the number of participants in the survey. Of all the general, nonlinear, increasing transformations that are also differentiable, only the linear transformation conforms to this requirement. In this case the back-transformed mean is equal to the sample mean.

Dimensional analysis has been used to show that where structural view dependence exists, one or more sensitivity functions will be dependent on one or more ratios of views, or, alternatively, on the ratios of one or more views to the back-transformed mean. Moreover, structural view dependence is possible as soon as the view of more than one person is sought. These findings are corroborated using the explicitly defined power transformation with a power, $b \neq 1$. Thus, for example, if $0 \leq b < 1$, corresponding to a concave, increasing function, then anyone's view that is larger than the back-transformed mean is downgraded below the equal weighting figure: $\partial Z/\partial X_i < 1/N$.

The approach required for the treatment of outliers in the sample has been discussed and shown to be fundamentally different from the case of physical measurements when it is the views of people that are being considered. While theory often exists to suggest that there is a single value for a physical parameter, no corresponding theory of uniqueness is likely to apply to the parameters chosen for opinion survey. There is thus no justification for downgrading a person's view based on the size of that view relative to the views of others in the sample. Once procedural errors (such as data entry faults) have been corrected, the only justification for downgrading or rejecting a person's view is that he is unfit to offer one. Such unfit persons are likely to have been screened out of the sample at an earlier stage, however. As a result, it is clear that no justification can be offered for trimmed means, nor for the median as representative of the sample and hence of the population when human views are being measured, because both are exclusionary and hence structurally view dependent.

With the exception of the linear transformation, all general, nonlinear, increasing and differentiable transformations are structurally view dependent. Any analyst using one of these can tell at the outset, without even consulting the data, whether his back-transformed mean will be less than or greater than the sample mean. Moreover, he will know that the reason for the discrepancy is that the method he is employing is systematically giving greater weight either to low views or to high views respectively. It is clear, therefore, that he cannot avoid the charge of lack of objectivity.

Out of the general, nonlinear, increasing and differentiable transformations, which include the power transformations as a subset, only the linear transformation, which produces the sample mean, satisfies the requirement of structural view independence. An analyst using the sample mean will be able to refute any charge of lack of objectivity in his consolidation of the various views in the sample into a single figure.

Although it is possible to postulate a transformation that is not covered within the transformations analysed in this paper, the range of transformations considered is nevertheless so wide that the onus must now be on the analyst putting forward a different transformation to show that it conforms to the requirement of structural view independence and is thus objective.

11. Acknowledgements

The author would like to record his gratitude to Roger Jones (Honorary Research Fellow, School of Engineering and Mathematical Sciences, City University London) for his invaluable help and advice during the writing of this paper.

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Appendix A. The relationship between logarithmic and power transformations

We begin by showing that

$$\ln x = \lim_{b \to 0} \frac{x^b - 1}{b} \qquad \text{(A.1)}$$

Putting

$$y = \frac{x^b}{b} \qquad \text{(A.2)}$$

and differentiating with respect to x gives:

$$\frac{dy}{dx} = \frac{x^b}{x} \qquad \text{(A.3)}$$

As $b \to 0$, so the right-hand side of equation (A.3) reduces to $\frac{1}{x}$. Since

$$\frac{d}{dx} \ln x = \frac{1}{x} \qquad \text{(A.4)}$$

it follows that

$$\frac{d}{dx} \ln x = \lim_{b \to 0} \frac{d}{dx} \frac{x^b}{b} \qquad \text{(A.5)}$$

Integrating both sides of equation (A.5) gives

$$\ln x = \lim_{b \to 0} \frac{x^b}{b} + k \qquad \text{(A.6)}$$

We may deduce the value of k from the behaviour at $x = 1$:

$$\ln 1 = 0 = \frac{1}{b} + k \qquad \text{(A.7)}$$

so that

$$k = -\frac{1}{b} \qquad \text{(A.8)}$$

Substituting from equation (A.8) into equation (A.6) produces equation (A.1). [The form of power transformation, $\left(x^b - 1\right)/b$, is described as "slightly preferable" in Box and Cox (1964). The same mathematical formula can arise in power utility functions, where $b = 1 - \varepsilon$, in which ε is risk-aversion (Thomas, 2010).]

The average of the logarithmically transformed variable may be written as both:

$$W = \frac{1}{N}\sum_{i=1}^{N} \ln X_i \qquad \text{(A.9)}$$

and, using equation (A.1):

$$W = \lim_{b \to 0}\left(\frac{1}{Nb}\sum_{i=1}^{N} X_i^b - \frac{1}{b} \right) \qquad \text{(A.10)}$$

Carrying out the reverse transformation on equation (A.9) gives:

$$Z = \exp(W) = \exp\left(\frac{1}{N}\sum_{i=1}^{N} \ln X_i \right) \qquad \text{(A.11)}$$

while applying the reverse power transformation (see equation (6)) to equation (A.10) gives:

$$Z = \lim_{b \to 0}\left(\frac{1}{N}\sum_{i=1}^{N} X_i^b \right)^{\frac{1}{b}} \qquad \text{(A.12)}$$

It is clear from the form of equation (A.12) that making the simple power transformation, $y = x^b$, then averaging and transforming back would give the same result. Hence, as far as back-transformed means are concerned, the log transformation may be regarded as equivalent to the simple power transformation, but for the special case that $b \to 0$.

The equivalence of the right-hand side of equations (A.11) and (A.12) allows us also to write down the further result:

$$\exp\left(\frac{1}{N}\sum_{i=1}^{N} \ln X_i \right) = \lim_{b \to 0}\left(\frac{1}{N}\sum_{i=1}^{N} X_i^b \right)^{\frac{1}{b}} \qquad \text{(A.13)}$$

where $\exp\left(\left(\sum_{i=1}^{N} \ln X_i\right) / N\right)$ is, of course, the geometric mean.

Appendix B. The back-transformed mean from the arc tan transformation when $N = 2$

When $N = 2$, equations (25) and (26) combine to give:

$$\begin{aligned} Z &= \tan\left(\frac{\alpha_1}{2} + \frac{\alpha_2}{2}\right) \\ &= \frac{\tan\left(\frac{\alpha_1}{2}\right) + \tan\left(\frac{\alpha_2}{2}\right)}{1 - \tan\left(\frac{\alpha_1}{2}\right)\tan\left(\frac{\alpha_2}{2}\right)} \end{aligned} \qquad \text{(B.1)}$$

where the second line follows from a trigonometric identity. When $\alpha_1 = \alpha_2 = \alpha_i$, the same identity can be used to give:

$$\tan(\alpha_i) = \frac{2\tan\left(\frac{\alpha_i}{2}\right)}{1 - \tan^2\left(\frac{\alpha_i}{2}\right)} \qquad i = 1,2 \qquad \text{(B.2)}$$

which may be rearranged to give the quadratic in $\tan\left(\frac{\alpha_i}{2}\right)$:

$$\tan^2\left(\frac{\alpha_i}{2}\right) \frac{2}{\tan(\alpha_i)} \tan\left(\frac{\alpha_i}{2}\right) - 1 = 0 \qquad i = 1,2 \qquad \text{(B.3)}$$

for which the required, positive root is:

$$\begin{aligned} \tan\left(\frac{\alpha_i}{2}\right) &= \frac{\sqrt{1 + \tan^2(\alpha_i)} - 1}{\tan(\alpha_i)} \\ &= \frac{\sqrt{1 + X_i^2} - 1}{X_i} \end{aligned} \qquad i = 1,2 \qquad \text{(B.4)}$$

where the relation, $X_i = \tan\alpha_i$ (from equation (24)) has been used in the second line. Substituting from equation (B.4) into equation (B.1) produces the back-transformed mean as:

$$Z = \frac{X_2\sqrt{1 + X_1^2} + X_1\sqrt{1 + X_2^2} - X_1 - X_2}{X_1 X_2 + \sqrt{1 + X_1^2} + \sqrt{1 + X_2^2} - \sqrt{(1 + X_1^2)(1 + X_2^2)} - 1} \qquad \text{(B.5)}$$

Figures for "Structural view independence: a criterion for judging the objectivity of economic parameters measured by opinion survey"

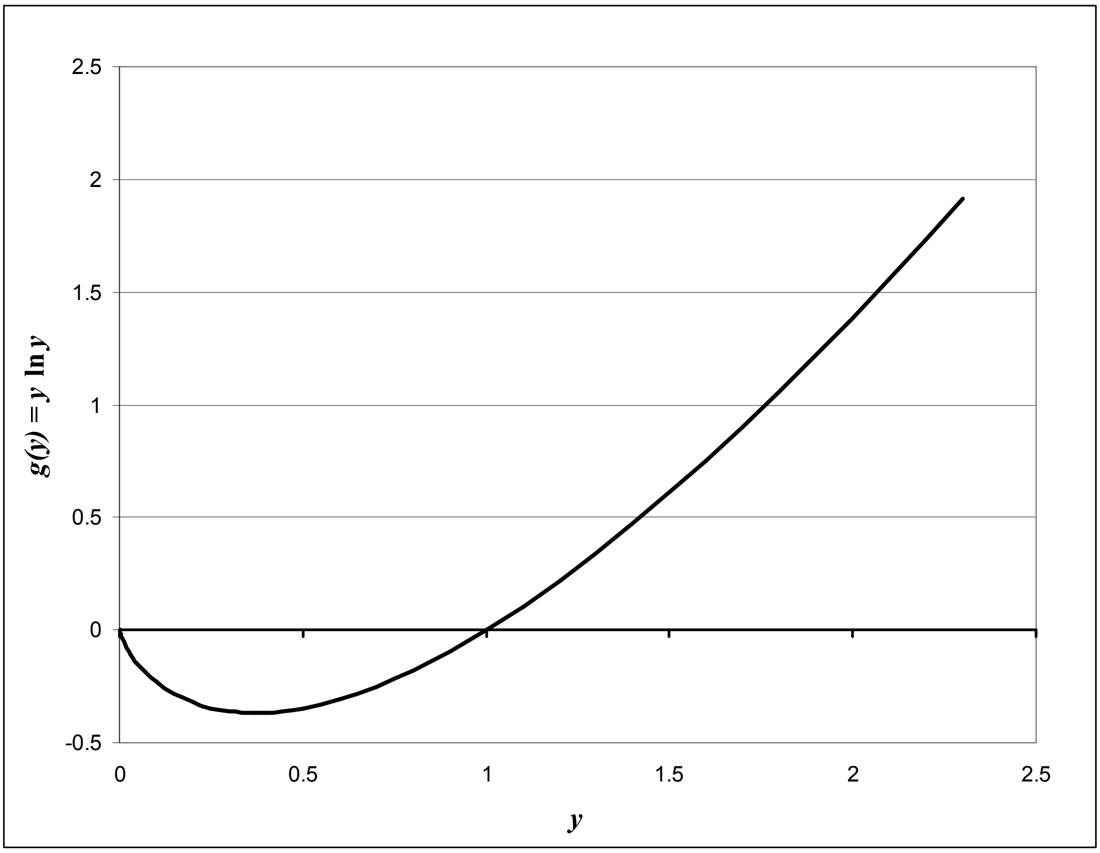

Figure 1. Demonstrating that the function, $g(y) = y \ln y$, is convex

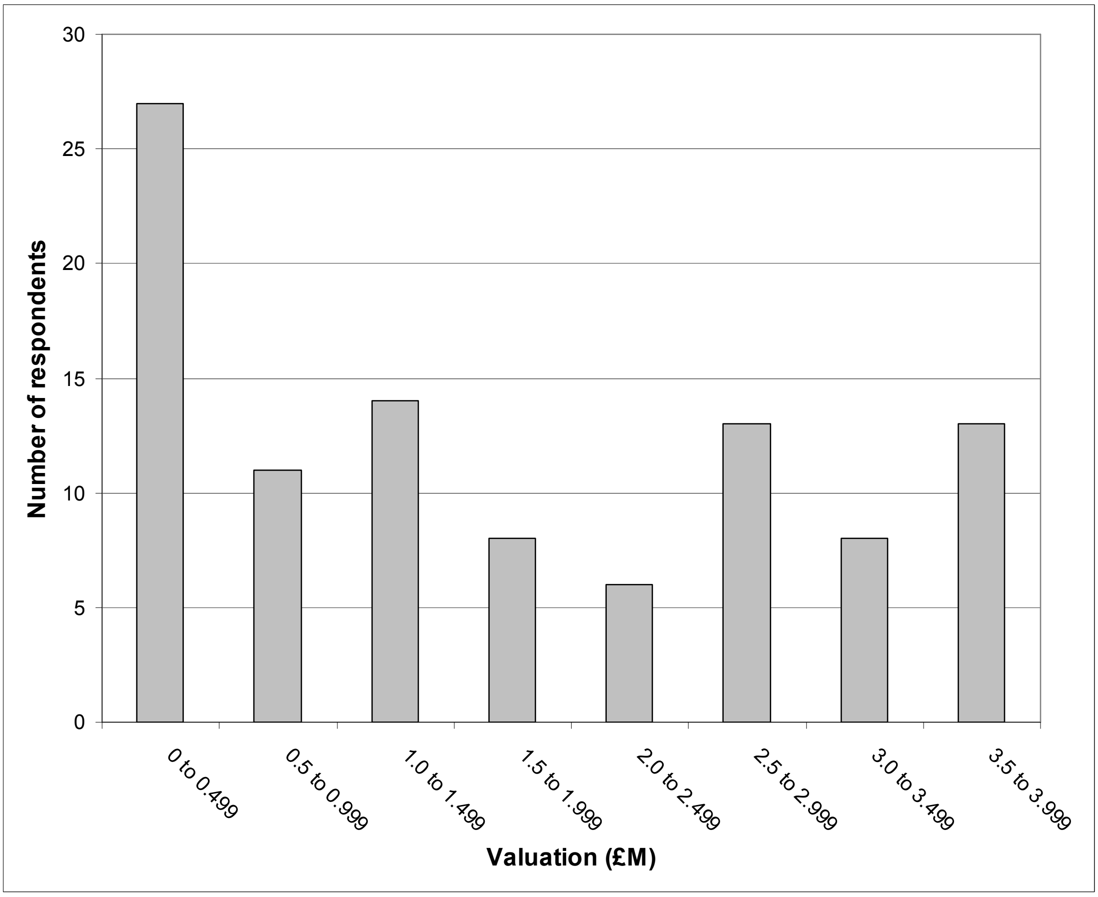

Figure 2. Distribution of valuation responses in the example of Section 2.2

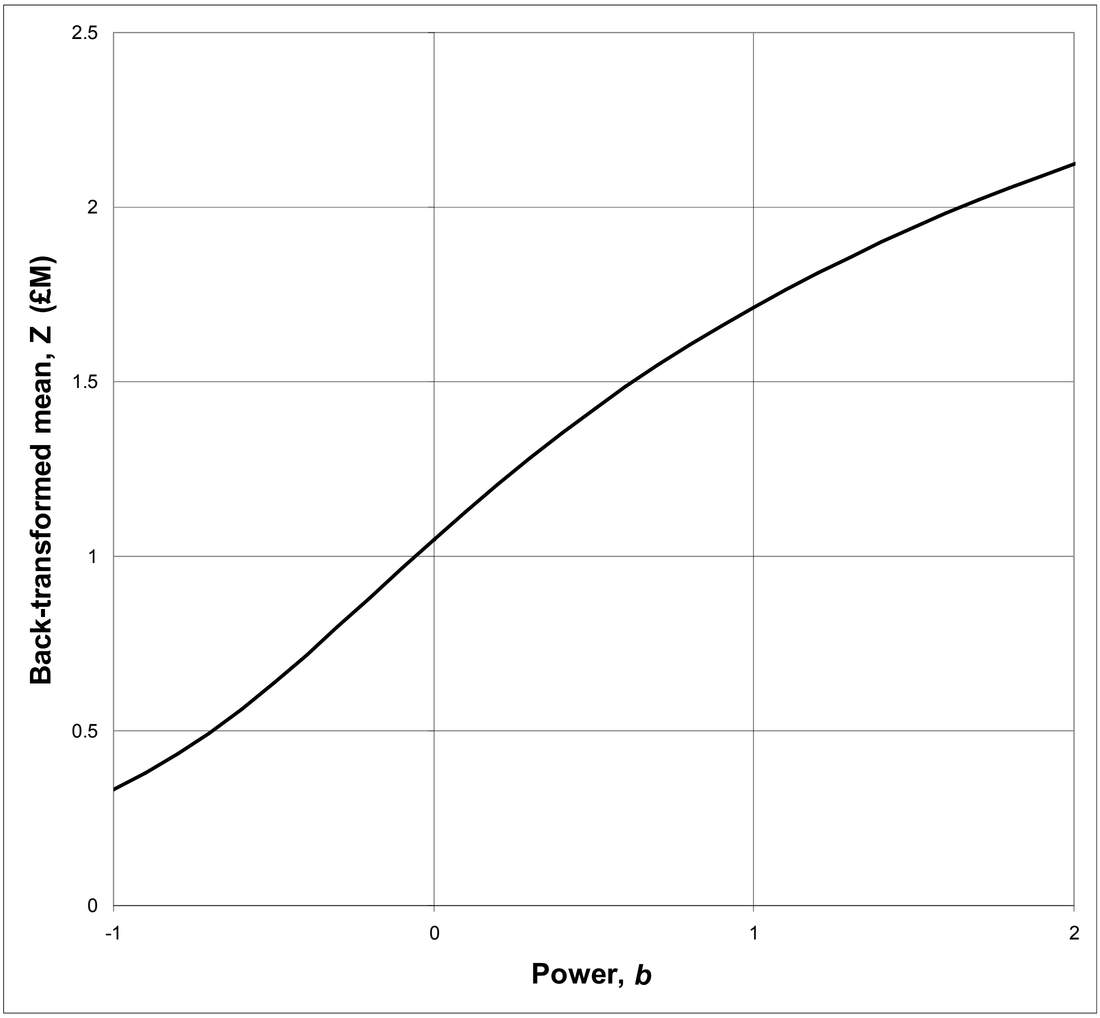

Figure 3. Back-transformed means for a range of powers, *b*. The geometric mean occurs at *b* = 0 and the sample mean at *b* = 1.

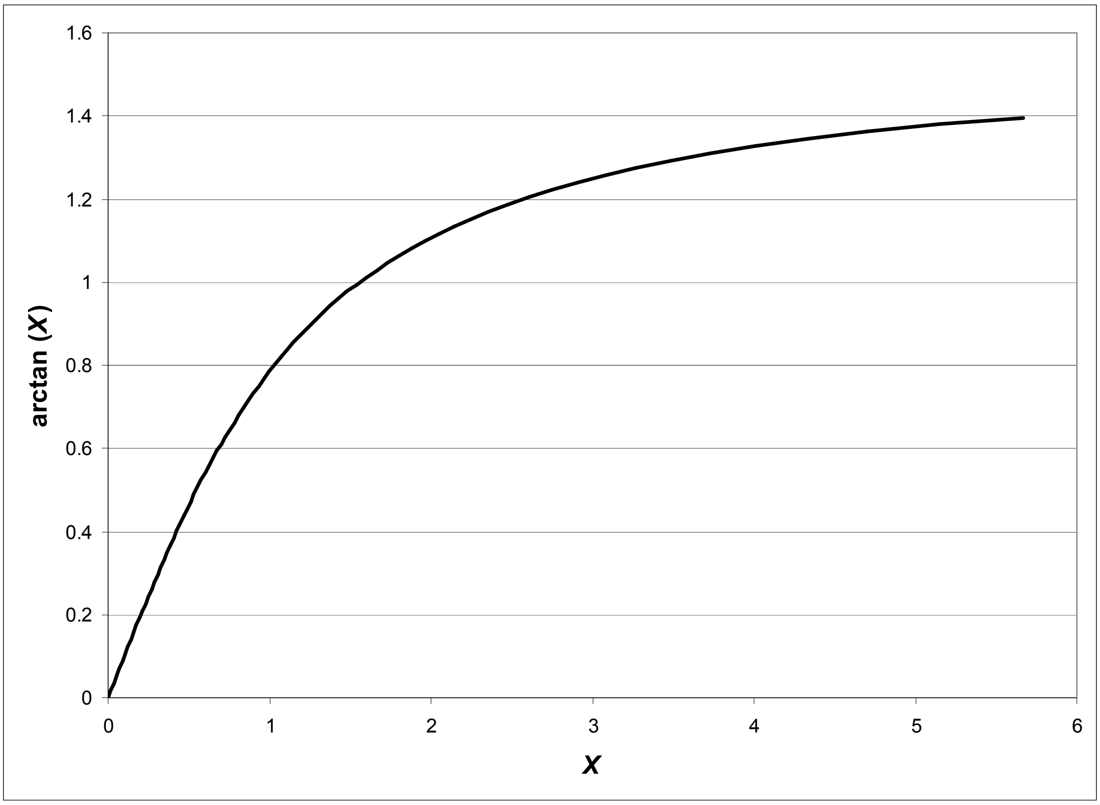

Figure 4. The arc tan transformation

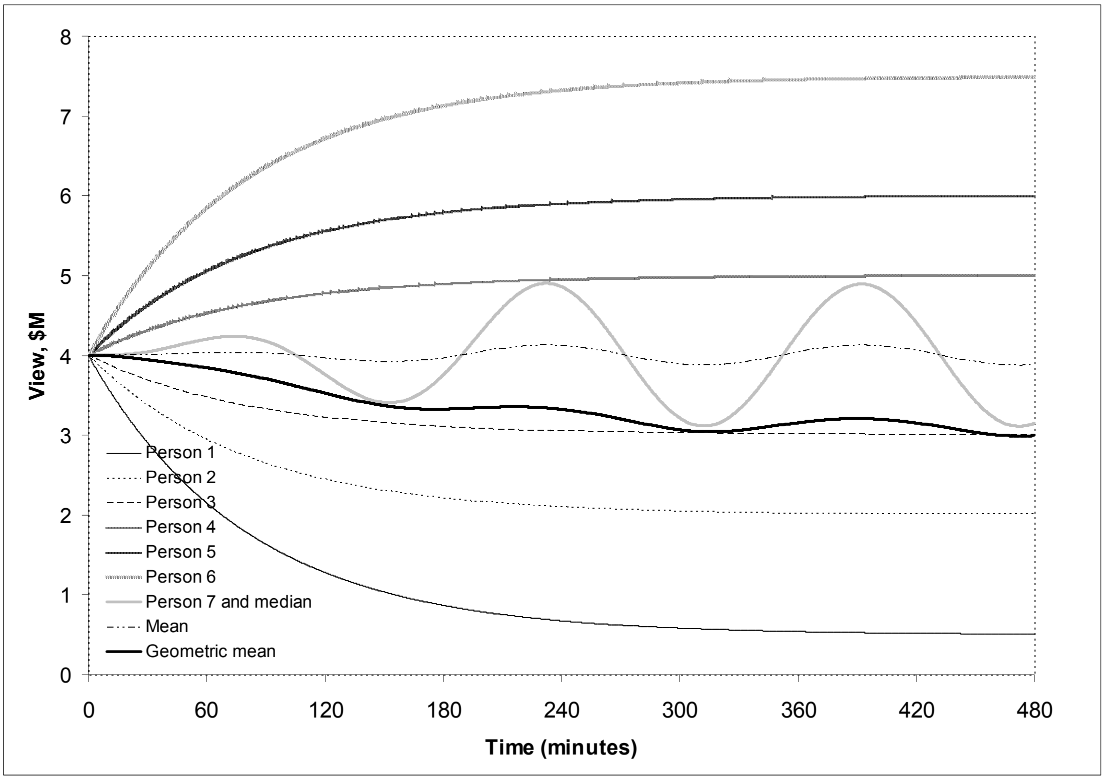

Figure 5. Section 9: the evolution of the views over the day

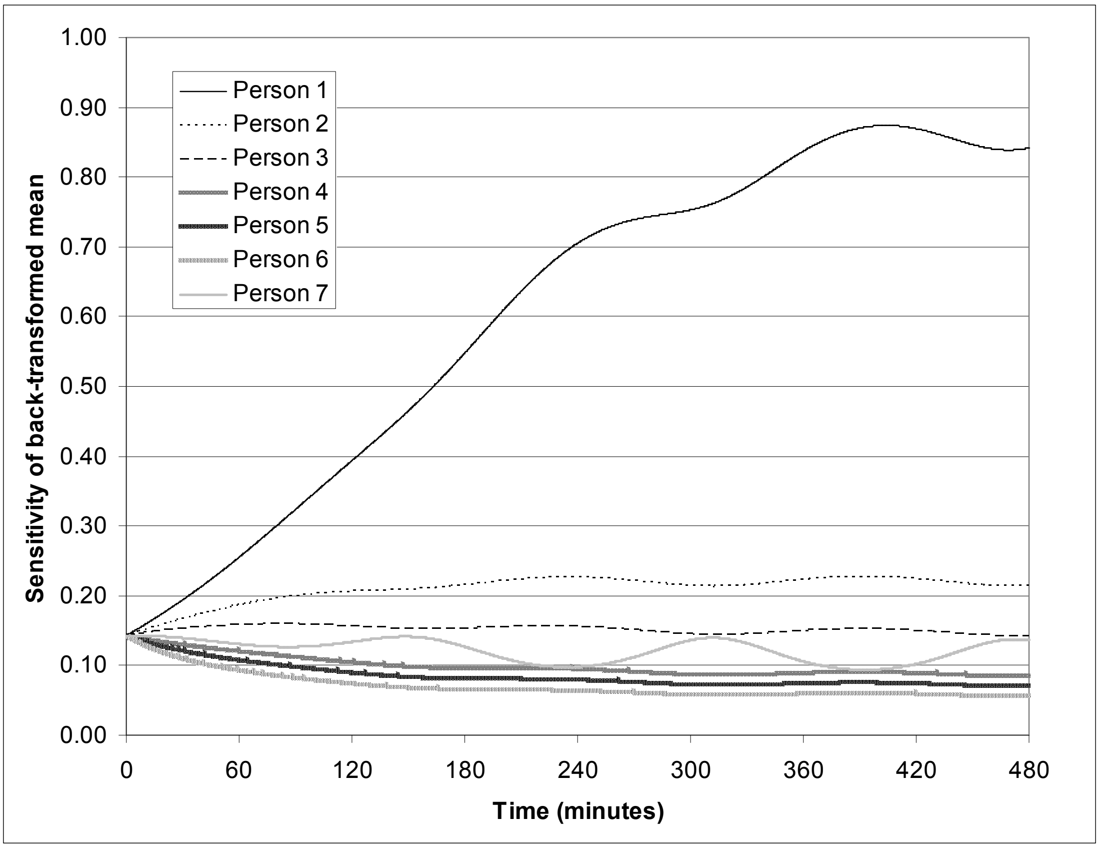

Figure 6. Section 9: the sensitivity of the back-transformed mean to the views of the people in the sample